RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2604

Pure Mathematics 4

Wednesday **9 JUNE 2004** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

[Turn over

1 A curve has equation $y = \dfrac{x(3x-10)}{(x-2)^2}$.

(i) Write down the equations of the two asymptotes. [2]

(ii) Find $\dfrac{dy}{dx}$. Hence find the coordinates of the stationary point. [5]

(iii) Sketch the curve. [3]

(iv) On a separate diagram, sketch the curve with equation $y = \left|\dfrac{x(3x-10)}{(x-2)^2}\right|$. [2]

(v) Solve the equation $\left|\dfrac{x(3x-10)}{(x-2)^2}\right| = 3$. [5]

(vi) Solve the inequality $\left|\dfrac{x(3x-10)}{(x-2)^2}\right| < 3$. [3]

2 (a) Prove by induction that, for every positive integer n, $(7^n + 3n + 8)$ is divisible by 9. [6]

(b) The complex number α is $-3 + \mathrm{j}$.

(i) Find α^2 and α^3. [3]

The complex number α is a root of the cubic equation $pz^3 + 22z^2 + qz + 40 = 0$, where p and q are real numbers.

(ii) Show that $p = 3$ and find q. [6]

(iii) Find the other two roots of the cubic equation. [5]

3 **(a)** Find the sum of the series

$$\frac{1}{1 \times 4} + \frac{1}{2 \times 5} + \frac{1}{3 \times 6} + \dots + \frac{1}{n(n+3)}.$$ [8]

(b) **(i)** The vectors **v** and **w** are non-parallel, and the angle between their directions is θ. Write down an expression for $|\mathbf{v} \times \mathbf{w}|$ in terms of $|\mathbf{v}|$, $|\mathbf{w}|$ and θ. [1]

Three points have coordinates $A(1, -8, -2)$, $B(9, 2, 4)$ and $C(-3, 2, 10)$.

(ii) Find the vector product $\overrightarrow{AB} \times \overrightarrow{AC}$. [3]

(iii) Deduce the area of the triangle ABC. [3]

The plane P contains the point B and is perpendicular to AB. The plane Q contains the point C and is perpendicular to AC.

(iv) Find the equation of the line of intersection of P and Q. [5]

4 The matrix $\begin{pmatrix} 3 & -2 \\ 4 & -6 \end{pmatrix}$ defines a transformation **M** of the (x, y) plane.

A triangle S has area 3 square units, and **M** transforms S to a triangle T.

(i) Find the area of T. [2]

(ii) Find the matrix which transforms T to S. [2]

(iii) Find the image of the line $y = 2x + 1$ under the transformation **M**. [3]

(iv) Find the two values of m for which $y = mx$ is an invariant line under the transformation **M**. [5]

The matrix $\begin{pmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{pmatrix}$ defines a reflection **R**.

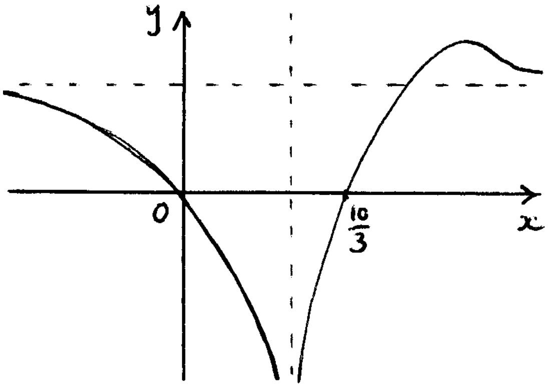
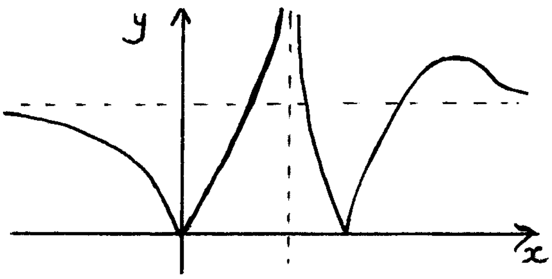
(v) By considering invariant points, or otherwise, find the equation of the mirror line of **R**. [3]

S_1 is the image of S under reflection in the line $y = -x$.

T_1 is the image of T under the reflection **R**.

(vi) Find the matrix which transforms S_1 to T_1. [5]

Mark Scheme

1 (i)	$x = 2, \quad y = 3$	B1B1	**2**	
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-2)^2(6x-10) - x(3x-10)2(x-2)}{(x-2)^4}$ $= \frac{2(10-x)}{(x-2)^3}$ For a stationary point, $2(10-x) = 0$ Stationary point is $(10, \frac{25}{8})$	M1 A1 M1 A1A1	**5**	Use of quotient rule (or equivalent) Any correct form Linear or quadratic equation (e.g. $-2x^2 + 24x - 40 = 0$) Max A1 if any 'extras' given, but ignore $(2, \infty)$ etc
(iii)		B1 B1 B1	**3**	Cutting x-axis at $0, \frac{10}{3}$ (only) LH branch correct shape RH branch correct shape
(iv)		B1 ft B1	**2**	Negative sections reflected and positive sections unchanged Two sharp points on x-axis
(v)	$\frac{x(3x-10)}{(x-2)^2} = 3 \Rightarrow 3x^2 - 10x = 3(x^2 - 4x + 4)$ $x = 6$ $\frac{x(3x-10)}{(x-2)^2} = -3 \Rightarrow 3x^2 - 10x = -3(x^2 - 4x + 4)$ $6x^2 - 22x + 12 = 0$ $x = \frac{2}{3}, \quad 3$	M1 A1 M1 A1A1	**5**	If M0 then B2 for one value correct B3 for two values correct B5 for three values correct *SR* Max 4 if $x = 2$ also given
	OR $[\mathrm{f}(x)]^2 = 9 \Rightarrow 12x^3 - 116x^2 + 288x - 144 = 0$ M1 $4(x-3)(x-6)(3x-2) = 0$ M1 $x = 3, \ 6, \ \frac{2}{3}$ A1A1A1			Obtaining a *cubic* equation Finding at least one factor
(vi)	$x < \frac{2}{3}, \quad 3 < x < 6$	M1 A1A1 ft	**3**	Considering intervals defined by three values (other than 2) found in (v) OR Both $\mathrm{f}(x) > -3$ and $\mathrm{f}(x) < 3$ rearranged and factorised If M0 then B1 (ft) for $x < \frac{2}{3}$ B1 (ft) for $3 < x < 6$ B3 (ft) for correct answer

2 (a)	When $n=1,\ 7^n+3n+8=7+3+8=18$ which is divisible by 9 Assuming 7^k+3k+8 is divisible by 9 $7^{k+1}+3(k+1)+8$ $=7(7^k+3k+8)-18k-45$ which is divisible by 9 True for $n=k \Rightarrow$ True for $n=k+1$ Hence true for all positive integers n	B1 M1 M1A1 M1 A1 **6**	Or when $n=0$ Stated or clearly implied *Dependent on previous M1M1A1M1*
(b)(i)	$\alpha^2=8-6\mathrm{j}$ $\alpha^3=(8-6\mathrm{j})(-3+\mathrm{j})=-24+26\mathrm{j}-6\mathrm{j}^2$ $=-18+26\mathrm{j}$	B1 M1 A1 **3**	 Or $(-3+\mathrm{j})^3=-27+27\mathrm{j}-9\mathrm{j}^2+\mathrm{j}^3$
(ii)	$p(-18+26\mathrm{j})+22(8-6\mathrm{j})+q(-3+\mathrm{j})+40=0$ Equating real parts, $-18p-3q+216=0$ imaginary parts, $26p+q-132=0$ Solving, $p=3,\ q=54$	M1 M1 A1 ft A1 ft M1 A1 **6**	 Equating real or imaginary parts to obtain an equation involving p and q ($p=3$ AG) *SR* If $p=3$ assumed, B1 for $q=54$ (Maximum M1B1 i.e. 2 out of 6)
(iii)	Another root is $-3-\mathrm{j}$	B1	*All marks in (iii) can be awarded in (ii)*
	$(z+3-\mathrm{j})(z+3+\mathrm{j})$ $=z^2+6z+10$ $3z^3+22z^2+54z+40=(z^2+6z+10)(3z+4)$ Other root is $-\frac{4}{3}$	M1 A1 M1 A1 **5**	 $z=-\frac{4}{3}$ always earns 4 marks
	OR $(-3+\mathrm{j})+(-3-\mathrm{j})+\beta=-\frac{22}{3}$ M2 $-6+\beta=-\frac{22}{3}$ A1 $\beta=-\frac{4}{3}$ A1		
	OR $(-3+\mathrm{j})(-3-\mathrm{j})\beta=-\frac{40}{3}$ M2 $10\beta=-\frac{40}{3}$ A1 $\beta=-\frac{4}{3}$ A1		
	Alternative for (ii) and (iii) Another root is $-3-\mathrm{j}$ B1 $(z+3-\mathrm{j})(z+3+\mathrm{j})=z^2+6z+10$ M1A1 Cubic is $(z^2+6z+10)(pz+4)$ M1 Coeff of z^2, $6p+4=22$ M2A1 $p=3$ A1 Coeff of z, $10p+24=q$ M1 $q=54$ A1 Other root is $-\frac{4}{3}$ A1		

3 (a)	$\dfrac{1}{r(r+3)}=\dfrac{1}{3}\left(\dfrac{1}{r}-\dfrac{1}{r+3}\right)$	M1 A1	Finding partial fractions
	Sum is $\dfrac{1}{3}\left(\dfrac{1}{1}-\dfrac{1}{4}\right)+\dfrac{1}{3}\left(\dfrac{1}{2}-\dfrac{1}{5}\right)+\dfrac{1}{3}\left(\dfrac{1}{3}-\dfrac{1}{6}\right)+\ldots$	M1	Using partial fractions (in at least two terms)
	$\ldots+\dfrac{1}{3}\left(\dfrac{1}{n-1}-\dfrac{1}{n+2}\right)+\dfrac{1}{3}\left(\dfrac{1}{n}-\dfrac{1}{n+3}\right)$	A1 ft M1	Three terms correct 3 fractions left at beginning *or* end
	$=\dfrac{1}{3}\left(1+\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{n+1}-\dfrac{1}{n+2}-\dfrac{1}{n+3}\right)$	A1A1 ft	Condone r instead of n
	$=\dfrac{11}{18}-\dfrac{1}{3}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+\dfrac{1}{n+3}\right)$	A1 **8**	
	$=\dfrac{11}{18}-\dfrac{3n^2+12n+11}{3(n+1)(n+2)(n+3)}$		
	$=\dfrac{n(11n^2+48n+49)}{18(n+1)(n+2)(n+3)}$		
(b)(i)	$\lvert\mathbf{v}\times\mathbf{w}\rvert=\lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert\sin\theta$	B1 **1**	Condone inclusion of $\lvert\hat{\mathbf{n}}\rvert$
(ii)	$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}8\\10\\6\end{pmatrix}\times\begin{pmatrix}-4\\10\\12\end{pmatrix}$	B1	For $\begin{pmatrix}8\\10\\6\end{pmatrix}$ and $\begin{pmatrix}-4\\10\\12\end{pmatrix}$
	$=\begin{pmatrix}60\\-120\\120\end{pmatrix}$	M1 A1 **3**	Evaluating vector product *Ignore subsequent working*
(iii)	Area is $\frac{1}{2}(\mathrm{AB})(\mathrm{AC})\sin\theta=\frac{1}{2}\left\lvert\overrightarrow{AB}\times\overrightarrow{AC}\right\rvert$	M1	
	$=\frac{1}{2}\sqrt{60^2+120^2+120^2}$	M1	Evaluating $\left\lvert\overrightarrow{AB}\times\overrightarrow{AC}\right\rvert$
	$=90$	A1 **3**	
	OR $\theta=52.1°$ M1 Area is $\frac{1}{2}(\mathrm{AB})(\mathrm{AC})\sin\theta$ M1 $=90$ A1		Finding θ (by any method) *Dependent on previous M1* Accept anything rounding to 90.0
(iv)	P is $4x+5y+3z=58$ Q is $-2x+5y+6z=76$	M1 A1	Finding equation of P *or* Q Both correct
	$x=0 \Rightarrow y=8,\ z=6$ Direction is given by $\overrightarrow{AB}\times\overrightarrow{AC}$	M1 M1	Finding one point on intersection e.g. $(4, 0, 14)$, $(-3, 14, 0)$ Or other method to find direction
	OR Eliminating y, $6x-3z=-18$ M1 $z=2x+6$, $y=-2x+8$ M1		
	Equation is $\mathbf{r}=\begin{pmatrix}0\\8\\6\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\2\end{pmatrix}$	A1 **5**	Condone omission of '$\mathbf{r}=$' Accept ratio or parametric form

4(i)	$\det \mathbf{M} = -10$ Area of T is $3 \times 10 = 30$	M1 A1 **2**	-30 earns M1A0
(ii)	$\mathbf{M}^{-1} = \frac{1}{-10}\begin{pmatrix}-6 & 2\\ -4 & 3\end{pmatrix}$ $\left[= \begin{pmatrix}0.6 & -0.2\\ 0.4 & -0.3\end{pmatrix} \right]$	B1B1 **2**	For $\frac{1}{-10}$ and $\begin{pmatrix}-6 & 2\\ -4 & 3\end{pmatrix}$
(iii)	$\begin{pmatrix}3 & -2\\ 4 & -6\end{pmatrix}\begin{pmatrix}t\\ 2t+1\end{pmatrix}$ $= \begin{pmatrix}-t-2\\ -8t-6\end{pmatrix}$ Equation of image is $y = 8x + 10$	M1 A1 A1 **3**	*Or* transforming 2 pts on $y = 2x + 1$ *Does not need to be simplified* *Or* images of two points
(iv)	$\begin{pmatrix}3 & -2\\ 4 & -6\end{pmatrix}\begin{pmatrix}t\\ mt\end{pmatrix}$ $= \begin{pmatrix}3t-2mt\\ 4t-6mt\end{pmatrix}$ $4t - 6mt = m(3t - 2mt)$ $2m^2 - 9m + 4 = 0$ $m = \frac{1}{2},\ 4$	M1 A1 M1 A1A1 **5**	 M0 if m is on the wrong side
(v)	$\begin{pmatrix}-0.8 & 0.6\\ 0.6 & 0.8\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x\\ y\end{pmatrix}$ $-0.8x + 0.6y = x$ $[\ 0.6x + 0.8y = y\ ]$ $y = 3x$	M1 A1 A1 **3**	
	OR $\cos 2\theta = -0.8$ $[\ \sin 2\theta = 0.6\ ]$ M1 $2\theta = 2.498$ $(= 143°)$ $y = x\tan\theta$ where $\theta = 1.249$ A1 $y = 3x$ A1		
	OR Invariant lines are $y = 3x$ and $y = -\frac{1}{3}x$ Points on $y = 3x$ are invariant M1A1 Mirror line is $y = 3x$ A1		
	OR e.g. Point $\mathrm{P}(1, 0)$ has image $\mathrm{P}'(-0.8, 0.6)$ Gradient of PP' is $-\frac{1}{3}$ Gradient of mirror line is 3 M1A1 Mirror line is $y = 3x$ A1		
(vi)	Reflection in $y = -x$ has matrix $\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$ Required matrix is $\begin{pmatrix}-0.8 & 0.6\\ 0.6 & 0.8\end{pmatrix}\begin{pmatrix}3 & -2\\ 4 & -6\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$ $= \begin{pmatrix}0 & -2\\ 5 & -6\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$ $= \begin{pmatrix}2 & 0\\ 6 & -5\end{pmatrix}$	B1 M2 A1 ft A1 **5**	 Give M1 if order wrong Or $\begin{pmatrix}-0.8 & 0.6\\ 0.6 & 0.8\end{pmatrix}\begin{pmatrix}2 & -3\\ 6 & -4\end{pmatrix}$

Examiner’s Report

2604: Pure Mathematics 4

General Comments

The overall standard of work was very good. Most candidates wrote clear solutions, showing their arguments. The majority were able to complete three questions, although there were signs of haste in several scripts. Some candidates were only able to produce two good questions (scoring 2 or 3 out of 20 on the third) and others were only able to score highly on question 1. The candidates appeared to find this quite a difficult paper, and some of the work on induction, vectors and matrices was clearly unfamiliar. There were a surprising number of careless numerical and algebraic errors. About 15% of candidates scored 50 marks or more (out of 60), but about a quarter scored less than 25 marks. Question 1 was attempted by almost every candidate, and question 4 was by far the least popular.

Comments on Individual Questions

Q.1 This was the best answered question, with half the attempts scoring 15 marks or more (out of 20) and about 20% scoring full marks.

(i) Most candidates gave the two asymptotes correctly, although the horizontal one did cause some difficulty.

(ii) The quotient rule was usually applied correctly to find $\frac{\mathrm{d}y}{\mathrm{d}x}$. Some candidates made sign errors and some forgot to square the denominator. Rather more errors were made in the manipulation to find the stationary point, and sometimes $(2,\ 0)$ was also stated to be a stationary point.

(iii) The curve was very often sketched well. When marks were lost it was usually for the right-hand branch; candidates were expected to identify the intercept on the x-axis as $\frac{10}{3}$ and to show the curve approaching the horizontal asymptote from above.

(iv) The great majority of candidates knew exactly what to do here and reflected the negative sections of their curve, although the points on the x-axis were not always as sharp as they should have been. A few drew the graph of $y^2 = \mathrm{f}(x)$ instead (where $\mathrm{f}(x) = \frac{x(3x-10)}{(x-2)^2}$).

(v) Most candidates solved the equations $\mathrm{f}(x) = 3$ and $\mathrm{f}(x) = -3$ separately, and the correct answer was frequently obtained. However, many considered only one of these equations. Others tried to find all the solutions by solving $[\mathrm{f}(x)]^2 = 9$; this led to a cubic equation and was much more prone to error.

(vi) The solution to the inequality was often just written down, using the graph in part (iv) and the critical values from part (v). When this was done the answer was usually correct. Many other methods were used, such as solving $[\mathrm{f}(x)]^2 < 9$. Those who solved $\mathrm{f}(x) < 3$ and $\mathrm{f}(x) > -3$ separately were rarely careful enough when combining their answers. A great deal of time was probably wasted here by candidates starting again instead of using their previous results.

Q.2 (a) Most candidates verified the result for $n=1$ and considered $u_{k+1}=7^{k+1}+3(k+1)+8$, but made no further progress; this earned 2 out of the 6 marks. There were several correct proofs, almost always based on considering $u_{k+1}-7u_k$. Many considered $u_{k+1}-u_k=6\times7^k+3$ but very few of these could explain convincingly why $6\times7^k+3$ must be divisible by 9.

(b) By contrast, this part question was often handled very competently. The great majority of candidates calculated α^2 and α^3 correctly in part (i). Most tackled part (ii) by substituting in their values and equating real and imaginary parts; this was frequently done efficiently and accurately, although some assumed that $p=3$ instead of showing it. In part (iii) the fact that the complex conjugate was also a root was almost universally known, and a variety of methods were used to find the real root. The most common was to find the quadratic factor $(z^2+6z+10)$, which sometimes appeared as (z^2+6z+8) or (z^2+10). Some considered the product, or the sum, of the three roots. The root was quite often given as -4 instead of $-\frac{4}{3}$. Some candidates combined parts (ii) and (iii) by finding the quadratic factor first and writing the equation as $(z^2+6z+10)(pz+4)=0$, then equating coefficients to find *p*, *q*, and the real root.

Q.3 This was the worst answered question, with an average mark of about 9.

(a) Most candidates realised that they should find partial fractions and apply the difference method, and this was often performed accurately, although errors were sometimes made in the final three terms. An answer such as $\frac{11}{18}-\frac{1}{3(n+1)}-\frac{1}{3(n+2)}-\frac{1}{3(n+3)}$ is perfectly acceptable here; many candidates combined the algebraic fractions, or all four fractions, but they did not earn any more marks for this extra effort. There were some candidates who thought that the sum could be found as the reciprocal of $\sum r(r+3)$.

(b) Parts (i) and (iii) appeared to be unfamiliar work for very many candidates, and were badly answered. It was surprising how often $\cos\theta$ occurred instead of $\sin\theta$ in part (i), and the answer given was often a vector involving $\hat{\mathbf{n}}$ instead of a scalar. When the answer to part (i) was wrong, the simple method for finding the area in part (iii) was no longer obvious, and most candidates tried alternative methods. The most popular was using the scalar product to find θ and then using $\frac{1}{2}(AB)(AC)\sin\theta$, but many slips were made.

The vector product in part (ii) was usually calculated correctly, although there were a surprising number of numerical errors. The answer was often divided by the common factor of 60; this was condoned provided that the correct vector product had appeared somewhere in the working.

Part (iv) was often answered efficiently and correctly, finding the equations of the planes *P* and *Q*, finding one common point and using the vector product from part (ii) to give the direction of the line of intersection. Longer methods, such as finding two common points, were more prone to careless errors. A substantial number of candidates gave equations for *P* and *Q* which were in fact equations of lines, and they made no progress.

Q.4 This question was attempted by about a quarter of the candidates. It was often not completed, and the average mark was about 11.

In part (i) most candidates calculated the determinant (-10), but quite a few divided the area by 10 instead of multiplying. Some gave a negative answer.

In part (ii) most candidates realised that the inverse matrix was required, and found it correctly. There were occasional sign errors, particularly with the determinant.

In part (iii), those who transformed a general point on $y = 2x + 1$ were often unable to find the equation of the image line; those who transformed two particular points were more successful here.

Part (iv) was very well answered, with the majority of candidates obtaining the correct invariant lines.

Finding the mirror line in part (v) was clearly less familiar work, but those who searched for invariant points were usually successful. Very many candidates repeated the work in part (iv) with the new matrix; this gave two invariant lines, and further investigation was then needed to determine which of these was the mirror line.

Part (vi) was very often omitted. There were some correct solutions, but most made little progress. Common errors included an incorrect second reflection matrix, forming a product of two matrices instead of three, and multiplying the matrices in the wrong order.